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Interpolation of Convex Scattered Data in R^3 Based upon an Edge Convex Minimum Norm Network

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In this paper a characterization of the optimal (using the minimum norm criterion) interpolant, convex along the edges of a triangulation, using data at the vertices is obtained. We thereby generalize results obtained by Nielson for the unconstrained case.

1. INTRODUCTION

In a paper by Nielson [18] a method for interpolation of scattered data in R^3 is presented. More precisely, given data $(x_i, y_i, z_i) \in R^3$, $i = 1, ..., n$, a bivariate function S with continuous first order partial derivatives and with the property $S(x_i, y_i) = z_i$, $i = 1, ..., n$, is constructed. The method of Nielson consists of three separate steps:

(i) *Triangulation*. The points $V_i := (x_i, y_i) \in R^2$ are used as vertices of a triangulation of a domain in R^2. The papers of Lawson [15] and Akima [1] contain a good discussion of many aspects of triangulating the convex hull of V_i, $i = 1, ..., n$.

(ii) *Construction of the minimum norm network*. We pay special attention to this part of Nielson's method in our Section 2. We give a new

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proof of his result by using a different approach since this is necessary for the purposes of this paper.

(iii) *Blending*. S is extended to the entire domain by means of a blending method. A detailed description of this step can be found in Nielson [18, 19].

The paper by Nielson [18] as well as an extension to a case with splines under tension by Nielson and Franke [20] generalizes well known classical results for so-called best interpolation of data in R^2 (see, e.g., C. De Boor and R. Lynch [5], C. De Boor [6]).

Recently many other results generalizing different aspects of best interpolation of data in R^2 have appeared. We refer to these as shape preserving best interpolation. The term shape reflects properties like convexity, monotonicity, and/or positivity of the interpolant, see, e.g., [14, 16, 2, 3] and the recent survey [13] by Greiner.

The area of surface fitting combined with shape-preserving interpolation has also attracted the attention of several researches. We mention, e.g., the papers by Beatson and Ziegler [4], Carlson and Fritsch [7], Dodd, McAllister, and Roulier [12, 21], Schmidt [22], and Costantini and Fontanella [8]. Utreras and Varas in [24] combined the notion of best interpolation with monotonicity of the interpolant.

The present paper addresses the problem of characterizing the minimum norm interpolant such that it is convex along the grid lines of a triangular net. We will refer to such a function as an edge convex network (a more detailed definition is given in Section 3). The formulation and the proof of the main result, Theorem 3.1, are given in Section 3. Here we apply recent results on convex optimization in Hilbert spaces [17, 9, 10]. We also pay attention to the problem of characterizing convex data over a triangulation.

By Theorems 3.1 and 3.2 the minimum norm solution is obtained as the solution of a nonlinear system of equations. In Section 3 we also formulate a Newton type algorithm for solving this system and present some numerical examples.

Remark 1.1. The important problem of extending the edge convex network into a C^1-function which is convex everywhere is very hard for general convex data. In fact it seems possible to construct examples such that the edge convex network cannot be extended. See also [8, p. 488].

2. The Unconstrained Minimization Problem

Let $n \geqslant 3$ be a given integer and $V_i = (x_i, y_i) \in R^2$, $i = 1, \dots, n$, given data points. A *triangulation* T will consist of a collection of non-overlapping, non-degenerate closed triangles T_{ijk} with vertices at V_i, V_j, and V_k, such that no vertex of one triangle lies on the edge of another triangle.

We denote by I_t the set of triples of indices that determine the triangulation T. Let

$$D = \bigcup_{ijk \in I_t} T_{ijk}$$

and let ∂D be the boundary of D. For $ijk \in I_t$ denote by e_{ij} the edge going from V_i to V_j. Introduce the edge sets,

$$N_e := \{ij \mid i, j \in \{\alpha, \beta, \gamma\}, \alpha\beta\gamma \in I_t\}, \qquad E = \bigcup_{ij \in N_e} e_{ij}.$$

The oriented angle between e_{ij} and e_{ik} will be denoted by $\angle(e_{ij}, e_{ik})$.

Let F be a function of two variables defined on E. This function may be described by the following family of univariate functions,

$$f_{ij}(t) := F((1 - t/\|e_{ij}\|)\, V_i + tV_j/\|e_{ij}\|) = F(x_{ij}(t), y_{ij}(t)), \tag{2.1}$$

with

$$x_{ij}(t) = (1 - t/\|e_{ij}\|)\, x_i + tx_j/\|e_{ij}\|, \qquad y_{ij}(t) = (1 - t/\|e_{ij}\|)\, y_i + ty_j/\|e_{ij}\|,$$
$$0 \leqslant t \leqslant \|e_{ij}\|, \qquad \|e_{ij}\| := \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}, \quad ij \in N_e.$$

The family of functions

$$F = \{f_{ij}\}_{ij \in N_e}$$

will be called a *curve network*. We now introduce some function classes on the set E of edges:

$$L^2(E) = \{F = \{f_{ij}\}_{ij \in N_e} : f_{ij} \in L^2(0, \|e_{ij}\|)\}$$
$$W^2(E) = \{F = \{f_{ij}\}_{ij \in N_e} : f_{ij}, f'_{ij}, f''_{ij} \in L^2(0, \|e_{ij}\|)\}$$

and

$$C(E) = \{F = \{f_{ij}\}_{ij \in N_e} : F \in W^2(E),\ F = H|_E,\ H \in C^1(R^2),$$
$$H(x_i, y_i) = z_i,\ i = 1, ...n\}.$$

The smoothness assumption on $C(E)$, that F is the restriction to E of some function in $C^1(R^2)$, should be understood in the following equivalent way. If we consider the set of functions $\{f_{ij}\}_{ij \in N_e}$ as curves in R^3, then in every point V_i they have a common tangent plane.

For $F \in W^2(E)$ let us consider the functional

$$\sigma(F) = \sigma(\{f_{ij}\}_{ij \in N_e}) = \sum_{ij \in N_e} \int_0^{\|e_{ij}\|} [f''_{ij}]^2 \, dt.$$

For a given triangulation and given observation points $\{z_i\}$ we now consider, following Nielson [18], the extremal problem:

$$\textit{Find } F \in C(E) \textit{ such that } \sigma(F) = \inf_{F \in C(E)} \sigma(F). \qquad (\mathbf{P})$$

THEOREM 2.1. *Problem* (**P**) *has a unique solution* $F \in C(E)$.

Proof. Let $\{F_n\}_1^\infty \subset C(E)$ be a minimizing sequence. By selecting a subsequence we may assume that $f''_{ij,n} \rightharpoonup f''_{ij}$ weakly in $L^2(0, \|e_{ij}\|)$ as $n \to \infty$. Here $f_{ij,n} \in W^2(0, \|e_{ij}\|)$ and $f_{ij,n}(0) = z_i$, $f_{ij,n}(\|e_{ij}\|) = z_j$. Further for $f_{ij,n}$, $f'_{ij,n}$ we have convergence for all t, in particular for $t = 0$ and $t = \|e_{ij}\|$, i.e., at the points (x_k, y_k). Therefore the limit functions f_{ij} satisfy the same restrictions on the tangent curves as $f_{ij,n}$, i.e., $F = \{f_{ij}\}_{ij \in N_e} \in C(E)$. Since $\sigma(F)$ is a quadratic functional it is also clear that F is a minimizer.

Next, if F_0 and $F_1 \in C(E)$ are minimizers, then we may conclude, using that $C(E)$ is convex and that the functional $\int_E g^2 \, dt$ is strictly convex in $L^2(E)$, that $f''_{ij,0}(t) = f''_{ij,1}(t)$ for all $t \in [0, \|e_{ij}\|]$. Since $f_{ij,0}$ and $f_{ij,1}$ are equal at the end points, we conclude that $f_{ij,0}(t) = f_{ij,1}(t)$ for all t, and we have proved uniqueness. ∎

The solution of problem (**P**) will be denoted by F and refered to as the *minimum norm network*.

In [18] a complete characterization of the solution of (**P**) is given, i.e., a method is described which enables one to find F for any data (x_i, y_i, z_i), $i = 1, ..., n$, and any (allowable) triangulation T of $V_i = (x_i, y_i)$, $i = 1, ..., n$. This method reduces the problem to solving a system of linear equations for the unknowns $\partial F/\partial x(V_i)$ and $\partial F/\partial y(V_i)$, $i = 1, ..., n$. Similarly to the problem of best interpolation of data in R^2, the solution is a smooth interpolating curve network $F = \{f_{ij}\}_{ij \in N_e} \in C(E)$ for which the functions $\{f_{ij}\}$ are cubic polynomials.

For the purposes of this paper we shall apply a different view. The idea is to construct a system of simple curve networks, partially linear on the edges, and to represent the second derivative of the solution of (**P**) as a linear combination of the elements of this system.

The notations and definitions in the following are illustrated in Fig. 2.1. Let us denote

$$N_i := \{ij \mid e_{ij} \text{ is an edge in } E \text{ with starting point } V_i\}, \qquad i = 1, ..., n. \qquad (2.2)$$

Let m_i be the number of elements in N_i. Clearly $m_i \geqslant 2$ for $i = 1, ..., n$ and $m_i = 2$ only occurs for certain vertices lying on the boundary ∂D.

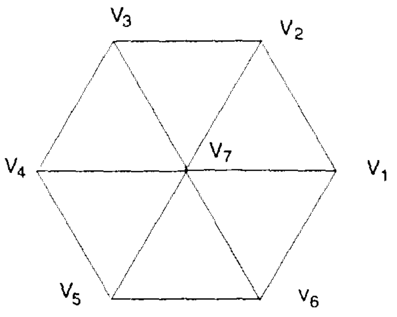

FIGURE 2.1.

In the subsequent construction of a basic network only those sets for which $m_i \geqslant 3$ will enter. In order to avoid repeating the condition $m_i > 2$ we adopt the convention that the list of indices $1, 2, ..., m_i - 2$ will simply be empty if $m_i = 2$. Now for such a set N_i we define an order of its elements. The first element ii_1 is chosen arbitrary. The $(k+1)$st element is defined such that $ii_{k+1} \in N_i$, $ii_{k+1} \neq ii_r$, $r = 1, ..., k$ and $\angle(e_{ii_k}, e_{ii_{k+1}})$ is minimal. In this way N_i becomes an ordered set

$$N_i = \{ii_1, ..., ii_{m_i}\}. \tag{2.3}$$

For the ordered sets N_i, defined by (2.2) and (2.3) we define linear networks on E called *basic networks*. Although this may be done in several ways we shall pursue only one possibility.

For a fixed i and for each $s = 1, ..., m_i - 2$ consider the following linear system in the unknowns $\lambda_{1,i}^{(s)}, \lambda_{2,i}^{(s)}, \lambda_{3,i}^{(s)}$,

$$\begin{gathered}
\lambda_{1,i}^{(s)} \frac{x_{i_s} - x_i}{\|e_{ii_s}\|} + \lambda_{2,i}^{(s)} \frac{x_{i_{s+1}} - x_i}{\|e_{ii_{s+1}}\|} + \lambda_{3,i}^{(s)} \frac{x_{i_{s+2}} - x_i}{\|e_{ii_{s+2}}\|} = 0 \\
\lambda_{1,i}^{(s)} \frac{y_{i_s} - y_i}{\|e_{ii_s}\|} + \lambda_{2,i}^{(s)} \frac{y_{i_{s+1}} - y_i}{\|e_{ii_{s-1}}\|} + \lambda_{3,i}^{(s)} \frac{y_{i_{s+2}} - y_i}{\|e_{ii_{s+2}}\|} = 0 \\
\lambda_{1,i}^{(s)} + \lambda_{2,i}^{(s)} + \lambda_{3,i}^{(s)} = 1.
\end{gathered} \tag{2.4}$$

The determinant of (2.4) is equal to 0 if and only if $\hat{e}_{ii_s}, \hat{e}_{ii_{s+1}}$, and $\hat{e}_{ii_{s+2}}$ (where $\hat{e}$ denotes a unit vector) are collinear which is impossible since $\{e_{ii_q}\}$, $q = s$, $s+1$, $s+2$ are three different edges in E with a common vertex. Hence (2.4) always has a unique solution.

If we assume that two of the numbers, e.g., $\lambda_{1,i}^{(s)}$ and $\lambda_{2,i}^{(s)}$ equal zero, then it follows $V_{i_{s+2}} \equiv V_{i_s}$. Therefore at least two of the numbers $\lambda_{1,i}^{(s)}, \lambda_{2,i}^{(s)}, \lambda_{3,i}^{(s)}$ are nonzero. It is easy to see that we can choose ii_1 so that

$$\lambda_{1,i}^{(s)} > 0, \qquad s = 1, ..., m_i - 2. \tag{2.5}$$

Moreover, if $V_i \notin \partial D$ then *any* choice of the initial ii_1 implies that $\lambda_{1,i}^{(s)} \neq 0$, $s = 1, 2, ..., m_i - 2$.

In this way for every $N_i = (ii_1, ..., ii_{m_i})$ we find $m_i - 2$ triples of numbers $\lambda_{r,i}^{(s)}$. For each of these triples we construct functions $\{B_{is}\}$, $s = 1, ..., m_i - 2$ on E such that each B_{is} has support only on three edges $e_{i,q}$, $q = i_s$, i_{s+1}, i_{s+2} where $iq \in N_i$. We take

$$B_{is} := \begin{cases} \lambda_{r,i}^{(s)} (1 - t/\|e_{iq}\|) & \text{on } e_{iq}, q = i_s, i_{s+1}, i_{s+2}, 0 \leqslant t \leqslant \|e_{iq}\| \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Here $r = r(q) = r(i_{s+j}) = j + 1$, $j = 0, 1, 2$.

Using the particular ordering $\{ii_1, ..., ii_{m_i}\}$ chosen for the sets N_i we have thus obtained a network $\{B_{is}\}$, $1 \leqslant i \leqslant n$, $1 \leqslant s \leqslant m_i - 2$ of functions on E. For later use let us also define functions B_{is} for $s = m_i - 1$ and m_i by solving (2.4) with $m_i + 1$ and $m_i + 2$ replaced by 1 and 2, respectively, so that, e.g., $e_{ii_{m_i+1}} := e_{ii_1}$.

Let $F, G \in C(E)$. For curve networks we define an inner product as

$$\langle F'', G'' \rangle = \sum_{ij \in N_e} \int_0^{\|e_{ij}\|} f''_{ij} g''_{ij} \, dt$$

so that $\sigma(F) = \langle F'', G'' \rangle$.

Consider the set of functions $\{f_{ij}\}_{ij \in N_e}$ as curves in R^3, cf. (2.1),

$$(x_{ij}(t), y_{ij}(t), f_{ij}(t)).$$

Then the vector $\mathbf{t}_{ij} \in R^3$,

$$\mathbf{t}_{ij} := \left(\frac{x_j - x_i}{\|e_{ij}\|}, \frac{y_j - y_i}{\|e_{ij}\|}, f'_{ij}(0) \right)$$

is a tangent vector at the point V_i. Note again that the definition of $C(E)$ requires that the vectors $\mathbf{t}_{ii_1}, \mathbf{t}_{ii_2}, ..., \mathbf{t}_{ii_{m_i}}$ are coplanar.

LEMMA 2.1. $\mathbf{t}_{ii_s}, \mathbf{t}_{ii_{s+1}}, \mathbf{t}_{ii_{s+2}}$ *are coplanar if and only if*

$$\lambda_{1,i}^{(s)} f'_{ii_s}(0) + \lambda_{2,i}^{(s)} f'_{ii_{s+1}}(0) + \lambda_{3,i}^{(s)} f'_{ii_{s+2}}(0) = 0, \tag{2.7}$$

where $\lambda_{1,i}^{(s)}, \lambda_{2,i}^{(s)}, \lambda_{3,i}^{(s)}$ *are defined by* (2.4).

Proof. Assume first that (2.7) holds. From (2.4) and (2.5) it follows that

$$\lambda_{1,i}^{(s)} \mathbf{t}_{ii_s} + \lambda_{2,i}^{(s)} \mathbf{t}_{ii_{s+1}} + \lambda_{3,i}^{(s)} \mathbf{t}_{ii_{s+2}} = 0$$

and the numbers $\lambda_{1,i}^{(s)}, \lambda_{2,i}^{(s)}, \lambda_{3,i}^{(s)}$ are not simulatenously equal to zero. Consequently the three vectors are linearly dependent, i.e., they are coplanar.

Next let $\mathbf{t}_{ii_s}, \mathbf{t}_{ii_{s+1}}, \mathbf{t}_{ii_{s+2}}$ be coplanar. Then there exist numbers $\mu_{i,1}^{(s)}$, $\mu_{2,i}^{(s)}, \mu_{3,i}^{(s)}$ that are not simultaneously zero and such that

$$\mu_{1,i}^{(s)}\mathbf{t}_{ii_s} + \mu_{2,i}^{(s)}\mathbf{t}_{ii_{s+1}} + \mu_{3,i}^{(s)}\mathbf{t}_{ii_{s+2}} = 0.$$

Then we have

$$\begin{aligned} &\mu_{1,i}^{(s)}\frac{x_{i_s}-x_i}{\|e_{ii_s}\|} + \mu_{2,i}^{(s)}\frac{x_{i_{s+1}}-x_i}{\|e_{ii_{s+1}}\|} + \mu_{3,i}^{(s)}\frac{x_{i_{s+2}}-x_i}{\|e_{ii_{s+2}}\|} = 0 \\ &\mu_{1,i}^{(s)}\frac{y_{i_s}-y_i}{\|e_{ii_s}\|} + \mu_{2,i}^{(s)}\frac{y_{i_{s+1}}-y_i}{\|e_{ii_{s+1}}\|} + \mu_{3,i}^{(s)}\frac{y_{i_{s+2}}-y_i}{\|e_{ii_{s+2}}\|} = 0 \\ &\mu_{1,i}^{(s)}f'_{ii_s}(0) + \mu_{2,i}^{(s)}f'_{ii_{s+1}}(0) + \mu_{3,i}^{(s)}f'_{ii_{s+2}}(0) = 0. \end{aligned} \tag{2.8}$$

From the first two equations of (2.8) and from (2.4) it follows that there is a constant $C \neq 0$ such that

$$\mu_{1,i}^{(s)} = C\lambda_{1,i}^{(s)}, \qquad \mu_{2,i}^{(s)} = C\lambda_{2,i}^{(s)}, \qquad \mu_{3,i}^{(s)} = C\lambda_{3,i}^{(s)}.$$

Then from the last equation of (2.8) it follows

$$C(\lambda_{1,i}^{(s)}f'_{ii_s}(0) + \lambda_{2,i}^{(s)}f'_{ii_{s+1}}(0) + \lambda_{3,i}^{(s)}f'_{ii_{s+2}}(0)) = 0.$$

Since $C \neq 0$, then $\lambda_{1,i}^{(s)}f'_{ii_s}(0) + \lambda_{2,i}^{(s)}f'_{ii_{s+1}}(0) + \lambda_{3,i}^{(s)}f'_{ii_{s+2}}(0) = 0.$ ∎

Next we present a Peano-type lemma.

LEMMA 2.2. *$F \in C(E)$ if and only if for every i, $1 \leqslant i \leqslant n$ it holds*

$$\langle F'', B_{is}\rangle = d_{is}, \tag{2.9}$$

where

$$d_{is} = \frac{\lambda_{1,i}^{(s)}}{\|e_{ii_s}\|}(z_{i_s} - z_i) + \frac{\lambda_{2,i}^{(s)}}{\|e_{ii_{s+1}}\|}(z_{i_{s+1}} - z_i) + \frac{\lambda_{3,i}^{(s)}}{\|e_{ii_{s+2}}\|}(z_{i_{s+2}} - z_i), \quad s = 1, ..., m_i - 2. \tag{2.10}$$

Proof. From the definition of B_{is}, $s = 1, ..., m_i - 2$, it follows

$$\begin{aligned} \langle F'', B_{is}\rangle = &\int_0^{\|e_{ii_s}\|} f''_{ii_s}(t)\,\lambda_{1,i}^{(s)}(1 - t/\|e_{ii_s}\|)\,dt \\ &+ \int_0^{\|e_{ii_{s+1}}\|} f''_{ii_{s+1}}(t)\,\lambda_{2,i}^{(s)}(1 - t/\|e_{ii_{s+1}}\|)\,dt \\ &+ \int_0^{\|e_{ii_{s+2}}\|} f''_{ii_{s+2}}(t)\,\lambda_{3,i}^{(s)}(1 - t/\|e_{ii_{s+2}}\|)\,dt. \end{aligned}$$

After integration by parts we have

$$\langle F'', B_{is}\rangle = -\lambda^{(s)}_{1,i} f'_{ii_s}(0) - \lambda^{(s)}_{2,i} f'_{ii_{s+1}}(0) - \lambda^{(s)}_{3,i} f'_{ii_{s+2}}(0)$$
$$+\frac{\lambda^{(s)}_{1,i}}{\|e_{ii_s}\|}(z_{i_s}-z_i)+\frac{\lambda^{(s)}_{2,i}}{\|e_{ii_{s+1}}\|}(z_{i_{s+1}}-z_i)+\frac{\lambda^{(s)}_{3,i}}{\|e_{ii_{s+2}}\|}(z_{i_{s+2}}-z_i).$$

Now, to establish sufficiency, let $F\in C(E)$ and therefore $\{f_{ij}\}_{ij\in N_e}$ have a common tangent plane in every point V_i, $i=1, ..., n$. That means for every such i the vectors $\mathbf{t}_{ii_s}$, $\mathbf{t}_{ii_{s+1}}$, $\mathbf{t}_{ii_{s+2}}$, $s=1, ..., m_i-2$ lie in the tangent plane. Then according to Lemma 2.1

$$\lambda^{(s)}_{1,i} f'_{ii_s}(0) + \lambda^{(s)}_{2,i} f'_{ii_{s+1}}(0) + \lambda^{(s)}_{3,i} f'_{ii_{s+2}}(0) = 0$$

and so from (2.10) we obtain (2.9).

For the necessity assume that (2.9) holds. It follows from (2.10) that

$$\lambda^{(s)}_{1,i} f'_{ii_s}(0) + \lambda^{(s)}_{2,i} f'_{ii_{s+1}}(0) + \lambda^{(s)}_{3,i} f'_{ii_{s+2}}(0) = 0, \qquad s=1, ..., m_i-2.$$

From Lemma 2.1 it follows that the vectors $\mathbf{t}_{ii_s}$, $\mathbf{t}_{ii_{s+1}}$, $\mathbf{t}_{ii_{s+2}}$ are coplanar for $s=1, ..., m_i-2$ and therefore $\mathbf{t}_{ii_1}, ..., \mathbf{t}_{ii_{m_i}}$ are coplanar as well. This shows $F\in C(E)$. ∎

We may now formulate,

THEOREM 2.2. *$F\in C(E)$ is the solution of* (**P**) *if and only if*

$$F'' = \sum_{i=1}^{n} \sum_{s=1}^{m_i-2} \alpha_{is} B_{is},$$

where $\alpha_{is}\in R$.

Proof. Problem (**P**) is equivalent to the problem

$$\text{find} \inf_{G\in L^2(E)} \int_E G^2\,dt, \tag{2.11}$$

when $\int_E G(t)\,B_{is}(t)\,dt = d_{is}$, $i=1, 2, ..., n$, $1\leqslant s\leqslant m_i-2$, and $G=\{g_{ij}\}_{ij\in N_e}$.

In fact, let

$$W^2(E)\supset Z=\{F=\{f_{ij}\}_{ij\in N_e}: f_{ij}(0)=z_i, f_{ij}(\|e_{ij}\|)=z_j\}$$

and consider the one-to-one mapping

$$Z\ni F=\{f_{ij}\}_{ij\in N_e}\mapsto \{g_{ij}\}_{ij\in N_e}=G\in L^2(E)$$

defined by $g_{ij} = f''_{ij}$. By Lemma 2.2, this mapping maps the set $C(E) \subset Z$ onto the submanifold

$$\left\{ G \in L^2(E) \colon \int GB_{is}\, dt = d_{is}\ 1 \leqslant i \leqslant n,\ 1 \leqslant s \leqslant m_i - 2 \right\} \subset L^2(E),$$

whence we conclude that F solves (**P**) if and only if G solves (2.11). Now by a well known Lagrange multiplier theorem applied to problem (2.11) the solutions G and F are of the form

$$F'' = G = \sum_{i=1}^{n} \sum_{s=1}^{m_i - 2} \alpha_{is} B_{is}$$

for some constants $\alpha_{is} \in R$, $1 \leqslant i \leqslant n$, $1 \leqslant s \leqslant m_i - 2$. ∎

LEMMA 2.3. *The functions* $\{B_{is}\}$, $1 \leqslant i \leqslant n$, $1 \leqslant s \leqslant m_i - 2$, *are linearly independent in* $L^2(E)$.

Proof. Assume that $\sum_{i,s} \alpha_{is} B_{is}(t) = 0$. Then by construction

$$f''_{ii_1}(0) = \alpha_{i1} B_{i1}(0) = \alpha_{i1} \lambda^{(1)}_{1,i} = 0.$$

From (2.5) it follows that $\alpha_{i1} = 0$. Further $f''_{ii_2}(0) = \alpha_{i1} \lambda^{(1)}_{2,i} + \alpha_{i2} \lambda^{(2)}_{1,i} = 0$ and therefore $\alpha_{i2} \lambda^{(2)}_{1,i} = 0$ and $\alpha_{i2} = 0$. Also

$$f''_{ii_3}(0) = \alpha_{i1} \lambda^{(1)}_{3,i} + \alpha_{i2} \lambda^{(2)}_{2,i} + \alpha_{i3} \lambda^{(3)}_{1,i} = 0 + \alpha_{i3} \lambda^{(3)}_{1,i} = 0,$$

and so $\alpha_{i3} = 0$. Continuing in a similar manner we obtain $\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{im_i - 2} = 0$. ∎

We immediately have the following corollary.

COROLLARY 2.1. $F \in C(E)$ *solves* (**P**) *if and only if* $F'' = \sum_{i=1}^{n} \sum_{s=1}^{m_i - 2} \alpha_{is} B_{is}$. *The coefficients* α_{is} *are obtained as the unique solution to the linear system of equations*

$$\sum_{i=1}^{n} \sum_{s=1}^{m_i - 2} \alpha_{is} \int_E B_{is} B_{kl}\, dt = d_{kl}, \qquad 1 \leqslant k \leqslant n, \qquad 1 \leqslant l \leqslant m_k - 2. \qquad (2.12)$$

The system matrix $\{\int B_{is} B_{kl}\}_{is,kl}$ *is symmetric and positive definite, provided that the same enumeration is used for the unknowns and the equations.*

Remark 2.1. For a corner where $m_i = 2$ we have $f''_{ij}(0) = 0$, $j = i_1, i_2$.

Remark 2.2. It is rather easy to see that $d_{kl} = 0$, $k = 1, 2, \ldots, n$, $l = 1, 2, \ldots, m_k - 2$ if and only if all the points $\{V_i, z_i\}_{i=1}^{n}$ are coplanar. In this case the interpolating curve network $\{f_{ij}\}_{ij \in N_e}$ that solves (**P**) consists of line segments only.

Remark 2.3. In the univariate case the functions B_{is} correspond to the well known hat functions (linear B-Splines) over the particular knot set. Also in the univariate case these hat functions are continuous contrary, in general, to the present case.

Remark 2.4. The number of the edges of T is $3(n-2)-(k-3)$, where k denotes the number of boundary edges. The number of the unknowns in the system (2.12) is

$$\begin{aligned}\sum_{i=1,\, m_i \geqslant 3}^{n} (m_i - 2) &= \sum_{i=1,\, m_i \geqslant 3}^{n} m_i - 2 \sum_{i=1,\, m_i \geqslant 3}^{n} 1 \\ &= \sum_{i=1}^{n} m_i - 2 \sum_{i=1}^{n} 1 = 2[3(n-2)-(k-3)] - 2n \\ &= 4n - 2k - 6.\end{aligned}$$

Since $k \geqslant 3$ then the number of unknowns does not exceed $4(n-3)$.

EXAMPLE. Let us consider the data obtained from a regular hexagonal pyramid, see Fig. 2.1. In this case $n = 7$ and

$$\begin{aligned} &V_1 = (1, 0) = -V_4, \qquad V_2 = (1/2, \sqrt{3}/2) = -V_5, \\ &V_3 = (-1/2, \sqrt{3}/2) = -V_6, \qquad V_7 = (0, 0). \end{aligned}$$

Also we take

$$z_i = 0, \qquad i = 1, ..., 6, \quad z_7 = -10.$$

Then

$$N_e = \{12, 23, 34, 45, 56, 61, 71, 72, 73, 74, 75, 76\},$$

and

$$\begin{aligned} &N_1 = \{12, 17, 16\}, \qquad N_2 = \{23, 27, 21\}, \qquad N_3 = \{34, 37, 32\}, \\ &N_4 = \{45, 47, 43\}, \qquad N_5 = \{56, 57, 54\}, \qquad N_6 = \{61, 67, 65\}, \\ &N_7 = \{71, 72, 73, 74, 75, 76\}, \\ &\qquad m_i = 3, \quad i = 1, ..., 6, \quad m_7 = 6, \quad \|e_{ij}\| = 1, \quad ij \in N_e. \end{aligned}$$

We define B_{is}, $i = 1, ..., 7$ by (2.6).

The unique triples of numbers that define B_{i1}, $i=1, \ldots, 6$ are

$$\lambda_{1,i}^{(1)}=1, \qquad \lambda_{2,i}^{(1)}=-1, \qquad \lambda_{3,i}^{(1)}=1, \qquad i=1, \ldots, 6.$$

For B_{7s}, $s=1, \ldots, 4$ we have

$$\lambda_{1,7}^{(s)}=1, \qquad \lambda_{2,7}^{(s)}=-1, \qquad \lambda_{3,7}^{(s)}=1, \qquad s=1, \ldots, 4.$$

To find the solution we have to solve the system of linear equations (2.12), i.e.,

$$\left\langle \sum_{i=1}^{7} \sum_{j=1}^{m_i-2} \alpha_{ij} B_{ij}, B_{kl} \right\rangle = d_{kl},$$

where $d_{kl}=10$, $k=1, \ldots, 6$, $l=1$; $k=7$, $l=1, \ldots, 4$. This system can be efficiently assembled taking into account the support properties of the functions B_{kl}. For the convenience of the reader we give the first equation. Since B_{11} has support only on the edges e_{12}, e_{17}, and e_{16} one easily finds, cf. Fig. 2.1, the first equation to be

$$\alpha_{11}+\alpha_{21}/6+\alpha_{61}/6-\alpha_{71}/6=d_{11}=10.$$

After some computations one obtains

$$\alpha_{i1}=12, \quad i=1, \ldots, 6, \qquad \alpha_{71}=\alpha_{74}=36, \qquad \alpha_{72}=\alpha_{73}=72.$$

Therefore

$$f_{12}''(t)=f_{23}''(t)=f_{34}''(t)=f_{45}''(t)=f_{56}''(t)=f_{61}''(t)=12, \qquad 0 \leqslant t \leqslant 1$$

and then

$$f_{12}(t)=f_{23}(t)=f_{34}(t)=f_{45}(t)=f_{56}(t)=f_{61}(t)=6t^2-6t, \qquad 0 \leqslant t \leqslant 1.$$

We have also

$$f_{7j}''(1)=-12 \qquad \text{and} \qquad f_{7j}''(0)=36, \quad j=1, \ldots, 6.$$

Then

$$f_{7j}''(t)=-48t+36, \qquad 0 \leqslant t \leqslant 1$$

and

$$f_{7j}(t)=-8t^3+18t^2-10, \qquad 0 \leqslant t \leqslant 1, \quad j=1, \ldots, 6.$$

Note that although the data are convex, the functions f_{7j}, $j=1, \ldots, 6$, are not convex.

3. The Constrained Minimization Problem

In this section we will assume that the triangulated domain $D = \bigcup_{ijk \in I} T_{ijk} \subset R^2$ is convex. Now, for a given triangulation, consider the function $l : D \to R$ with the property that l is continuous, linear on each triangle, and $l(V_i) = z_i$, for $1 \leqslant i \leqslant n$, i.e., *$l$ is the piecewise linear interpolant* of the data points.

Definition 3.1. The data (x_i, y_i, z_i), $1 \leqslant i \leqslant n$, are said to be convex with respect to a given triangulation if the function l is convex. The data are strictly convex if in addition the gradient of l has a jump discontinuity across each interior edge.

Note that for given data the function l may be convex with respect to one triangulation but not with respect to another as the following simple example shows. Let $(V_1, z_1) = (1, 0, 1)$, $(V_2, z_2) = (0, 1, 0)$, $(V_3, z_3) = (-1, 0, 1)$, and $(V_4, z_4) = (0, -1, 0)$. Consider the two triangulations,

$$T_1 = \{(V_1, V_2, V_4), (V_2, V_3, V_4)\}, \qquad T_2 = \{(V_1, V_2, V_3), (V_3, V_4, V_1)\}.$$

The data are convex w.r.t. T_1 but not w.r.t. T_2.

If the points z_i are samples of a strictly convex function then there exists a unique triangulation Δ such that the data are convex with respect to Δ. In this case the function l is the largest convex function which is minorant of the data values, i.e.,

$$l(x) = \sup\{L(x) : L : R^2 \to R,\ L \text{ linear},\ L(x_i, y_i) \leqslant z_i,\ \forall i\}.$$

The function l is then the piecewise linear convex interpolant. Compare also [11, Chap. 10; 23].

Next we introduce the following concept.

Definition 3.2. Every function in the set $C(E)$ whose restriction to each edge is convex will be called edge convex.

We shall in the following characterize and compute such an edge convex function of minimal norm and therefore we consider the problem,

($\mathbf{P}_c$) Find $F \in C(E)$, minimizing the functional

$$\int_E F''^2(t)\, dt$$

under the side condition $f''_{ij}(t) \geqslant 0$, for all $ij \in N_e$.

Arguing as in the previous section we see that $(\mathbf{P}_c)$ has the following equivalent formulation,

$(\mathbf{P}_c')$ Find $G \in L^2(E)$, minimizing the functional

$$\int_E G^2(t)\,dt$$

under the linear side conditions

$$\int_E G(t)\,B_{kl}(t)\,dt = d_{kl}, \qquad 1 \leqslant k \leqslant n, \quad 1 \leqslant l \leqslant m_k - 2$$

and under the side condition $G(t) \geqslant 0$.

Let us state the following theorem which is the main result of the paper. Here $(x)_+$ denotes the positive part of x.

THEOREM 3.1. *Assume that data are strictly convex. Then the problem* $(\mathbf{P}_c)$ *has a unique solution* F *where* F'' *is of the form*

$$F''(t) = \left(\sum_{i=1}^{n} \sum_{s=1}^{m_i - 2} \alpha_{is} B_{is}(t)\right)_+.$$

Moreover the coefficients α_{is} *are determined as a solution of the nonlinear system of equations*

$$\int_E \left(\sum_{i=1}^{n} \sum_{s=1}^{m_i - 2} \alpha_{is} B_{is}(t)\right)_+ B_{kl}(t)\,dt = d_{kl}, \qquad 1 \leqslant k \leqslant n, \quad 1 \leqslant l \leqslant m_k - 2. \tag{3.1}$$

We will prove Theorem 3.1 and therefore we need a few auxiliary lemmas and definitions. First we have,

LEMMA 3.1. *If the data are convex (strictly convex) then there exists a convex (strictly convex) function* $\psi \in C^\infty(R^2)$ *interpolating the points* (x_i, y_i, z_i), *i.e., such that* $\psi(x_i, y_i) = z_i$, $1 \leqslant i \leqslant n$.

Proof. Consider first the case when the data are strictly convex. Then for each point V_i, $1 \leqslant i \leqslant n$, there exists an affine function $L_i(x, y)$ such that $L_i(V_i) = l(V_i) = z_i$ and $L_i(x, y) < l(x, y)$ if $(x, y) \neq V_i$. Then there also exists a constant $c > 0$ such that

$$c\,|(x, y) - V_i|^2 + L_i(x, y) < l(x, y), \qquad \text{if} \quad (x, y) \neq V_i.$$

Now let $0 \leqslant \varphi \in C_0^\infty(R^2)$ be such that $\int_{R^2} \varphi(x, y)\, dx\, dy = 1$, $\int x\varphi\, dx\, dy = \int y\varphi\, dx\, dy = 0$ and take $\varphi_\varepsilon(x, y) = (1/\varepsilon^2)\, \varphi(x/\varepsilon, \; y/\varepsilon)$. Next define the function L by

$$L(x, y) = \max_{1 \leqslant i \leqslant n} \{L_i(x, y) + c\, |(x, y) - V_i|^2\}$$

so that $L(x, y) = L_i(x, y) + c\, |(x, y) - V_i|^2$ in some neighbourhood of each point V_i and $L < l$ if $(x, y) \neq V_i$. Consider the function $q = L * \varphi_\varepsilon$ ($*$ denotes convolution). By our assumptions we have

$$\begin{aligned} q(x, y) &= (c\, |(x, y) - V_i|^2 + L_i(x, y)) * \varphi_\varepsilon \\ &= c\, |(x, y) - V_i|^2 + L_i(x, y) + c \int (x^2 + y^2)\, \varphi_\varepsilon(x, y)\, dx\, dy \end{aligned}$$

in some neighbourhood of each V_i provided that $\varepsilon > 0$ is chosen small enough. Therefore we may take

$$\psi = q - c \int (x^2 + y^2)\, \varphi_\varepsilon(x, y)\, dx\, dy.$$

It is then clear that $\psi(V_i) = z_i$, $\psi \in C^\infty(R^2)$, $\psi \leqslant l$ and that ψ is strictly convex. Next, if the data are just convex, we may take $c = 0$ in the previous argument and the proof of the lemma is complete. ∎

To proceed we need to establish the connection between the signs of the second order divided differences d_{kl}, $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant m_k - 2$ as defined in (2.10) and the convexity of our data. We will subsequently assume, as in (2.5), that the sets N_i have been ordered so that $\lambda_{1,i}^{(s)} > 0$ for all i and s.

Let us now consider two adjacent faces $E_{ii_s} = \{(x, y, l(x, y)) : (x, y) \in T_{ii_s i_{s+1}}\}$ and $E_{ii_{s+1}} = \{(x, y, l(x, y)) : (x, y) \in T_{ii_{s+1} i_{s+2}}\}$ of the graph of l. Further let us introduce the associated triple product $(\mathbf{t}_{ii_s} \times \mathbf{t}_{ii_{s+1}}) \circ \mathbf{t}_{ii_{s+2}}$ where $\mathbf{t}_{ij}$ now is given by $\mathbf{t}_{ij} = ((x_j - x_i)/\|e_{ij}\|, \; (y_j - y_i)/\|e_{ij}\|, \; z_j - z_i)$. It is clear that the following lemma is valid.

Lemma 3.2. *The data are convex if and only if*

$$(\mathbf{t}_{ii_s} \times \mathbf{t}_{ii_{s+1}}) \circ \mathbf{t}_{ii_{s+2}} \geqslant 0$$

for $1 \leqslant i \leqslant n$, $1 \leqslant s \leqslant m_i$. *Similarly, the data are strictly convex, if and only if all these products are positive.*

Corollary 3.1. *The data are convex (strictly convex) if and only if* $d_{is} \geqslant 0$ $(d_{is} > 0)$ *for all* $i = 1, 2, \ldots, n$ *and* $s = 1, 2, \ldots, m_i$.

Proof. It follows from (2.10) and (2.4) that $d_{is}=0$ if and only if the vectors $\mathbf{t}_{ii_s}$, $\mathbf{t}_{ii_{s+1}}$, and $\mathbf{t}_{ii_{s+2}}$ are coplanar (the argument is as in the proof of Lemma 2.1). Next, using that $\lambda_{1,i}^{(s)}>0$, it is easy to see, e.g., by varying the value of z_{i_s} only, that $d_{is}>0$ if and only if $(\mathbf{t}_{ii_s}\times\mathbf{t}_{ii_{s+1}})\circ\mathbf{t}_{ii_{s+2}}>0$. ∎

Note that in this corollary s is assumed to run through all the values $1\leqslant s\leqslant m_i$ whereas in Theorem 2.2 and Theorem 3.1 only the values $s=1, 2, ..., m_i-2$ appear. For the proof of Theorem 3.1 we need a slightly sharper characterization of data convexity, using only basis functions B_{is}, $1\leqslant n$, $1\leqslant s\leqslant m_i-2$. Such a result is given in the following lemma.

LEMMA 3.3. *The ordering of the sets N_i, i.e., the way of defining the basis functions B_{is}, may be chosen in such a way that the data are convex (strictly convex) if and only if $d_{is}\geqslant 0$ $(d_{is}>0)$, $1\leqslant i\leqslant n$, $1\leqslant s\leqslant m_i-2$.*

Proof. Let us introduce the terminology that an edge e_{kr} is *covered* by the ordering of the sets $N_i=(ii_1, ii_2, ..., ii_{m_i})$ if it is true either that $r=k_v$ with $1<v<m_k$ or $k=r_v$ with $1<v<m_r$.

It suffices to prove that the ordering can be done in such way that all the edges $e_{kr}\not\subset\partial D$ are covered. In fact, if all these edges are covered, then for an arbitrary non-boundary edge e_{kr} either the condition $d_{kv-1}=\langle F'', B_{kv-1}\rangle>0$ or the condition $d_{rv-1}=\langle F'', B_{rv-1}\rangle>0$ enters, guaranteeing convexity across the edge e_{kr}.

Consider first the set ∂D of boundary points. For any $V_i\in\partial D$ we choose the ordering of N_i such that e_{ii_1} and $e_{ii_{m_i}}$ are the boundary edges. Then all non-boundary edges emanating from these points are covered.

Next consider the set W_1 of interior points which have some boundary point as a neighbor. For each $V_i\in W_1$ there exist V_j and $V_j\in\partial D$ such that the triangle $T_{ijk}\in D$. Choose the ordering of N_i such that $i_1=k$ and $i_{m_i}=j$. Then all the edges emanating from $V_i\in W_1$ are covered.

Let W_2 be the set of points not in $\partial D\cup W_1$ having some point in W_1 as a neighbour. For each $V_i\in W_2$ there exist V_j and $V_k\in W_1$ such that $T_{ijk}\in D$. Choose the ordering of N_i so that $i_1=k$ and $i_{m_i}=j$. Then all edges emanating from the points $V_i\in W_2$ are covered.

Proceeding in a similar way with sets W_3, W_4, ... until the point set $\{V_1, V_2, ..., V_n\}$ is exhausted, we obtain an ordering such that all interior edges are covered. ∎

Remark 3.1. If the condition that all edges are covered is violated then it is possible to construct an example which shows that the function I may be nonconvex although $d_{is}\geqslant 0$, $1\leqslant i\leqslant n$, $1\leqslant s\leqslant m_i-2$.

In order to proceed with the proof of Theorem 3.1 we will use some results from the theory of convex functionals in Hilbert spaces.

Let H be a Hilbert space, $C \subset H$ a closed, convex subset,

$$A\colon H \to R^N$$

a bounded, linear mapping and $y_0 \in R^N$ a fixed vector. $A^*\colon R^N \to H$ denotes the dual mapping. Consider the following minimization problem

(**P**$''$) Find $x \in H$, minimizing the functional $\|x\|^2$ under the side condition $Ax = y_0$ and $x \in C$.

It is clear that there exists a unique solution x, provided that the domain of definition $\{x \in H\colon Ax = y_0, x \in C\}$ is non-empty.

Theorems characterizing the solution of problem (**P**$''$) have been given, e.g., by Micchelli and Utreras [17] and Chui, Deutsch, and Ward [9]. A thorough analysis of this and similar problems is given in the two recent papers [9, 10]. See in particular Theorem 3.2 in [9], where we can find the the following result.

THEOREM 3.2. *Assume that* $y_0 \in \operatorname{int}(AC)$. *Then there exists a unique solution* x *of problem* (**P**$''$) *and* x *is of the form*

$$x = P_C(A^*y)$$

for some $y \in R^N$. *Conversely, if* $x = P_C(A^*y)$ *for some* $y \in H$ *and* $Ax = y_0$ *then* x *is the solution of* (**P**$''$).

Here P_C denotes the orthoprojection on the closed convex set C and AC is the set of image points $AC = \{y \in R^N\colon y = Ax \text{ for some } x \in C\}$. Further $\operatorname{int}(AC)$ is the interior of AC,

Returning to the problem (**P**$'_c$) we assume first that the definition of the functions B_{kl} has been made according to the construction in Lemma 3.3. Let the mapping

$$A\colon L^2(E) \to R^N, \qquad N = \sum_{k=1}^{n} (m_k - 2)$$

be defined by

$$L^2(E) \ni G \mapsto \left\{ \int_E G(t)\, B_{kl}(t)\, dt \right\} \in R^N,$$

$1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant m_k - 2$ and let $y_0 = \{d_{kl}\} \in R^N$. Then the dual mapping A^* has the form

$$R^N \ni \alpha = \{\alpha_{kl}\} \mapsto \sum_{k=1}^{n} \sum_{l=1}^{m_k-2} \alpha_{kl} B_{kl} = A^*\alpha = L^2(E).$$

Further, take $C = \{G \in L^2(E): G(t) \geqslant 0\}$. It is clear that A is linear and bounded and that $C \subset L^2(E)$ is closed and convex. It remains only to verify the condition that $y_0 = \{d_{kl}\}$ is an interior point of AC.

Now, assuming that the data are strictly convex, there exists by Lemma 3.1 a strictly convex function $\psi \in C^\infty(R^2)$ such that $\psi(x_i, y_i) = z_i$, $1 \leqslant i \leqslant n$. Let $F_0 = \psi|_E$ and $G_0 = F_0''$. Consequently $G_0 > 0$ on each edge e_{ij}, i.e., $G_0 \in C$ and by Lemma 2.2 we have $AG_0 = y_0$. Consider some neighbourhood $\{y: |y - y_0| < \varepsilon\}$ of y_0. If $\varepsilon > 0$ is small enough we have $y = \{c_{kl}\}$ with $c_{kl} > 0$ for each y in this neighbourhood. Further, by solving the linear system

$$AA^*\beta = y - y_0$$

we get $AG = y$ where $G = G_0 + A^*\beta$. Since $\inf_E G_0 > 0$ it is clear that $\inf_E G > 0$ if ε has been taken small enough. This proves that y_0 is an interior point of AC. By Theorem 3.2 we then conclude that $(\mathbf{P}_c')$ has a unique solution G. Since it is clear that $P_C(x) = x_+$ for every $x \in L^2(E)$ it follows that $(\mathbf{P}_c')$ has a unique solution of the form $G = (\sum_i \sum_s \alpha_{is} B_{is})_+$. Then it follows that $(\mathbf{P}_c)$ has a unique solution F with F'' of the form $F'' = (\sum_i \sum_s \alpha_{is} B_{is})_+$. It also follows that (3.1) is valid, and conversely, that any solution $\{\alpha_{is}\}$ of (3.1) gives the solution F of $(\mathbf{P}_c)$ via the equation $F'' = (\sum_i \sum_s \alpha_{is} B_{is})_+$. This representation of F'' has thus been established for the particular choice of a basic network $\{B_{is}\}$, $1 \leqslant i \leqslant n$, $1 \leqslant s \leqslant m_i - 2$, such that all edges are covered. However, with given strictly convex data the function $\sum_i \sum_s \alpha_{is} B_{is}(t)$ may be represented in any other permitted basic network $\{B'_{is}\}$ (since $\sum_i \sum_s \alpha_{is} B_{is}(t)$ represents the second derivative of a unique function in $C(E)$) so that

$$\sum_i \sum_s \alpha_{is} B_{is}(t) = \sum_i \sum_s \alpha'_{is} B'_{is}(t).$$

Therefore the statements in Theorem 3.1 are true for any basic network, provided that the data are strictly convex. This completes the proof of Theorem 3.1.

Note that Theorem 3.1 states that the function $F''(t) = (G(t))_+$, with

$$G(t) = \sum_{i=1}^{n} \sum_{s=1}^{m_i - 2} \alpha_{is} B_{is}(t),$$

is uniquely defined. In order to obtain a satisfactory theory we need to know whether also the coefficients $\{\alpha_{is}\}$ solving Eq. (3.1) are unique. This problem is resolved by the following theorem.

THEOREM 3.3. *If the coefficients* d_{is}, $1 \leqslant i \leqslant n$, $1 \leqslant s \leqslant m_i - 2$, *are given by strictly convex data, then Eq.* (3.1) *has a unique solution* $\{\alpha_{is}\}$.

Proof. It suffices to prove that the $\{\alpha_{is}\}$ are uniquely determined by the solution F via the equality

$$F''(t) = (G(t))_+ = \left(\sum_{i=1}^{n} \sum_{s=1}^{m_i - 2} \alpha_{is} B_{is}(t) \right)_+ .$$

Now consider a fixed point V_i. Since G is edge-wise linear we first conclude that if the restriction $G_+ |_{e_{is}} \not\equiv 0$ is known then $G|_{e_{is}}$ is uniquely determined. Secondly assume that $G|_{e_{is}} \not\equiv 0$ for all but at most two adjacent edges e_{iq} and e_{ir}. Without loss of generality we may take $e_{iq} = e_{i,m_i-1}$ and $e_{ir} = e_{i,m_i}$. Thus $G|_{e_{is}}$ is known for $1 \leqslant s \leqslant m_i - 2$.

Arguing as in the proof of Lemma 2.3 we conclude that $f''_{ii_1} = \alpha_{i1} B_{i1}(0)$ is uniquely defined, i.e., that α_{i1} is uniquely defined. Further $f''_{ii_2} = \alpha_{i1} \lambda^{(1)}_{2,i} + \alpha_{i2} \lambda^{(2)}_{1,i}$ is uniquely defined and therefore also α_{i2}. Continuing in a similar manner we obtain that $\alpha_{i1}, \alpha_{i2}, ..., \alpha_{im_i-2}$ are uniquely defined.

To conclude the proof it now suffices to verify the following lemma.

LEMMA 3.4. *G_+ can vanish on at most two edges e_{iq}, e_{is}. Further if $G_+ \equiv 0$ on two edges e_{iq} and e_{is} with $\angle(e_{iq}, e_{is}) \leqslant \pi$ then they are adjacent, i.e., there is no intermediate edge e_{ir}.*

Proof. For any triple of edges, e_{iq}, e_{ir}, e_{is} ordered counter clockwise we may consider the restriction $F|_{(e_{iq} \cup e_{ir} \cup e_{is})}$ and form the second divided difference $\bar{d}_{iq}$ and the basis function $\bar{B}_{iq}$ defined in the same way as in Section 2. Because of convexity it is true that

$$\bar{d}_{iq} = \langle G_+, \bar{B}_{iq} \rangle > 0$$

which implies that $G_+ > 0$ on at least one of the edges.

Next suppose that $G_+ \equiv 0$ on two edges e_{iq}, e_{is} with $\angle(e_{iq}, e_{is}) \leqslant \pi$ and e_{ir} is an intermediate edge. Then, considering the restriction of F to $e_{iq} \cup e_{ir} \cup e_{is}$, we have

$$\bar{\lambda}_{iq} > 0, \qquad \bar{\lambda}_{ir} \leqslant 0, \qquad \bar{\lambda}_{is} > 0$$

with these coefficients defined as in Eqs. (2.4) and (2.5). It follows that

$$\bar{d}_{iq} = \langle G_+, \bar{B}_{iq} \rangle \leqslant 0$$

which is a contradiction. ∎

Remark 3.2. The assumption that the domain $D = \bigcup_{ijk \in I_t}$ is convex is not necessary for the validity of the results in this section. It suffices to assume that there exists a strictly convex function $\psi \in C^\infty(R^2)$ (cf. Lemma 3.1) interpolating the points (x_i, y_i, z_i) and that the triangulation of D is a subset of the triangulation generated by the largest convex minorant l of the data values.

We end our paper by considering a few examples. The first is the example from Section 2, where now convexity is imposed as a shape constraint. Then the equations (3.1) become

$$\left\langle \left[\sum_{j=1}^{6} \alpha_{j1} B_{j1} + \sum_{s=1}^{4} \alpha_{7j} B_{7j} \right]_+ , B_{kl} \right\rangle = 10,$$

$$k = 7,\ l = 1, ..., 6; \qquad k = 1, ..., 6; \quad l = 1.$$

The solution is

$$\alpha_{71} = \alpha_{74} = 30(\sqrt{3}+4)/13 = \alpha_{72}/2 = \alpha_{73}/2$$
$$\alpha_{i1} = 30(7 + 5\sqrt{3})/13, \qquad i = 1, ..., 6$$

and

$$f_{12}(t) = f_{23}(t) = f_{34}(t) = f_{45}(t) = f_{56}(t) = f_{61}(t)$$
$$= -15(\sqrt{3}+4)\, t(1-t)/13, \qquad 0 \leqslant t \leqslant 1$$

and for $j = 1, ..., 6$,

$$f_{7j}(t) = \begin{cases} 5(6\sqrt{3}+11)(\sqrt{3}-1-t)^3/13 - 30(\sqrt{3}+4)(1-t)/13, \\ \qquad 0 \leqslant t \leqslant \sqrt{3}-1, \\ -30(\sqrt{3}+4)(1-t)/13, \\ \qquad \sqrt{3}-1 \leqslant t \leqslant 1. \end{cases}$$

TABLE 3.1

Data				Triangulation		
		V_i		V_6	V_1	V_2
i	x_i	y_1	z_i	V_6	V_2	V_3
1	−2	0	0	V_6	V_3	V_4
2	−1.6	0.2	−2	V_6	V_4	V_5
3	0	0.4	−3	V_6	V_1	V_2
4	1.6	0	−2.5	V_7	V_2	V_3
5	2	0	0	V_7	V_3	V_4
6	−0.5	2.3	−1.7	V_7	V_4	V_5
7	0.5	−2	−1.9	V_7	V_4	V_6

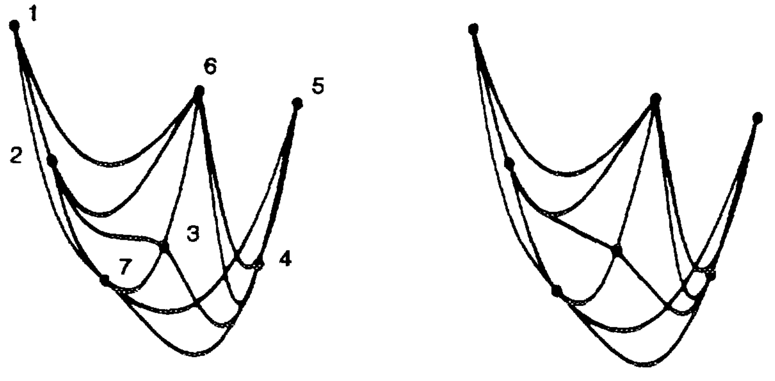

FIG. 3.1. Reconstructions based on Table 3.1. Left, unconstrained case. Right, constrained case.

In general, of course, the equations (3.1) cannot be solved explicitly. We propose to use Newton's method for its solution, similar as in the univariate case, [3]. It can be shown that Newton iteration becomes

$$\int_E \Big(\sum_{i,s} \alpha_{is}^{(r)} B_{is}(t)\Big)_+^0 \Big(\sum_{i,s} \alpha_{is}^{(r+1)} B_{is}(t)\Big) B_{kl}(t)\, dt = d_{kl}, \tag{3.2}$$

$k = 1, 2, ..., n$, $l = 1, 2, ..., m_{k-2}$. Here

$$(\varphi(t))_+^0 = \begin{cases} 1 & \text{if } \varphi(t) \geqslant 0, \\ 0 & \text{else.} \end{cases}$$

Remark 3.3. As in the univariate case [3] one may verify that the Jacobian occurring in Newton's method (3.2) is positive definite in a sufficiently small neighbourhood around α^*, the unique solution of the

TABLE 3.2

Data													
i	1	2	3	4	5	6	7	8	9	10	11	12	13
x_i	0.21	0.46	0.83	0.97	0.67	0.53	0.28	0.07	0.06	0.25	0.49	0.67	0.77
y_i	0.88	0.93	0.89	0.54	0.71	0.74	0.77	0.70	0.43	0.56	0.61	0.54	0.45
i	14	15	16	17	18	19	20	21	22	23	24	25	
x_i	0.90	0.66	0.50	0.32	0.25	0.46	0.57	0.75	0.94	0.46	0.18	0.14	
y_i	0.31	0.35	0.47	0.44	0.31	0.33	0.20	0.25	0.05	0.07	0.19	0.06	

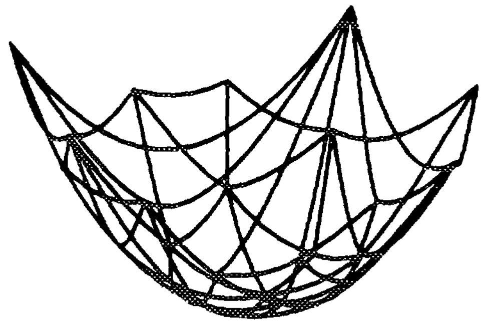

FIG. 3.2. Reconstruction based on Table 3.2.

equations (3.1). The (kl, is)th element of the Jacobian (evaluated at $\alpha = \alpha^{(r)}$) is

$$\int_E \left(\sum_{i,s} \alpha_{is}^{(r)} B_{is}(t) \right)_+^0 B_{is}(t) \, B_{kl}(t) \, dt.$$

The second example is given in Table 3.1 and the resulting unconstrained and constrained curve networks are displayed in Fig. 3.1. In the final example we consider the convex function

$$f = 5 * \exp((x - 0.5)^2 + (y - 0.5)^2)$$

sampled at 25 points as given in Table 3.2. The reconstruction is given in Fig. 3.2. Here we display only the constrained case since there were no visible differences between the unconstrained and the constrained case (although along some edges the constraint was active). This effect was quite typical in examples with many interpolation points.

In Newton's method we used the unconstrained solution as the starting value for the iterations. In the example from Fig. 3.1 (7 points) the 6th and 7th iterations are identical in double precision arithmetic. For the example from Fig. 3.2 (25 points) the 3rd and 4th iterations are identical.

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